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Sigma model corrections to the confining background

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Abstract: Sigma model ($\alpha\prime$) corrections to the confining string background are obtained. The main result is that the Poincaré invariant ansatz is maintained. Physical conditions for the dissapearance of the naked singularity are discussed.

Keywords: Confinement, Bosonic Strings.

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Contents

1. Introduction

In the paper [2] a new confining string background has been proposed. This background encodes in a precise way the renormalization group properties of the four dimensional gauge theory, and it guarantees the vanishing of the sigma model beta functions to $o(\alpha' \equiv l_s^2)$.

The explicit form of the background metric is:

$$ds^2 = g\,\eta_{\mu\nu} dx^\mu dx^\nu + l_c^2 dg^2 + \sum_{A=5}^{A=26} (dx^A)^2\,, \tag{1.1}$$

where l_c is a characteristic length, unrelated *a priori* with the *string scale*, l_s.

This metric is a particular instance of the Poincaré invariant family of metrics

$$ds^2 = a(\rho)\,\eta_{\mu\nu} dx^\mu dx^\nu + d\rho^2 + \sum_{A,B=5}^{26} g_{AB}(x^A) dx^A dx^B \tag{1.2}$$

for which several interesting holographic properties (such as a C-theorem and Callan-Symanzyk-like renormalization group equations) have been discussed in the literature [1].

The background dilaton reads

$$\Phi = -\log g \tag{1.3}$$

In terms of the usual renormalization group scale, μ, the dimensionless holographic variable is given by:

$$g = \frac{g_0}{\log\, \mu/\Lambda}\,, \tag{1.4}$$

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where Λ is the renormalization group invariant mass scale of the gauge theory. The ultraviolet (UV) region then corresponds to $g \ll 1$, whereas the infrared (IR) one lies in $g \gg 1$; by construction it does not make any sense to consider $\mu < \Lambda$, which would correspond to negative values of g and imaginary dilaton fields.

There is a question, however, about the range of the coordinates. We presumably want the 21 *spectator* coordinates to be compactified on a very small torus, of common radius, say, R.

On the other hand, the background has got a singularity at $g = 0$, but we obviously cannot trust it that far, and for some purposes it is better to consider the (extensible) manifold obtained by restriction to an interval $g \in (g_{UV}, g_{IR})$.

It is mathematically also possible to consider the solution extended to the real line, $g \in \mathbb{R}$, by symply putting absolute values:

$$ds^2 = |g|\eta_{\mu\nu}dx^\mu dx^\nu + l_c^2 dg^2 + \sum_{A=5}^{A=26}(dx^A)^2 \tag{1.5}$$

and

$$\Phi = -\log|g| \tag{1.6}$$

in terms of the physical renormalization group scale, μ, this means that we have extended μ to the interval $(0, \Lambda)$ by using

$$\mu \to \frac{\Lambda^2}{\mu}. \tag{1.7}$$

There are two physically different types of corrections to any physical quantity evaluated in a given background: sigma model and stringy. Stringy corrections are proportional to g_s, which for us is exactly the gauge coupling, g. Sigma model corrections are proportional to l_s^2, and are the subject of the present paper.

We expect roughly that sigma model corrections should become important when the curvature is big, as measured in l_s units, that is, for $R \sim 1/l_c^2 g^2 \gg 1/l_s^2$, that is, for $g \ll l_s^2/l_c^2$ (the UV region).

String corrections (to Green's functions or even to non local observables, such as Wilson loops), on the other hand, are expected to become big when $g \gg 1$ (the IR region). In fact both corrections are inextricably entangled through the soft dilaton theorem, as we have emphasized in previous work.

In the following, we are going to find that the solution above does indeed receive sigma model corrections; of course this is enough to prove that it is not a coset model, which would instead be an exact solution to the sigma model equations to all orders in l_s^2/l_c^2. We have nevertheless included a direct proof in the appendix, putting the emphasis on coset models which in the semiclassical approximation do enjoy the full Poincaré group as their isometry group.

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2. l_s^4 corrections

So far we know that the string background given by the formulae

$$
\begin{aligned}
ds^2 &= g\eta_{\mu\nu}dx^\mu dx^\nu + l_c^2 dg^2 + \sum_{A=5}^{A=26}(dx^A)^2 \\
\Phi &= -\log g
\end{aligned}
\tag{2.1}
$$

satisfies the beta functions to $O(\alpha')$.However we easily find that this solution is not exact. The Weyl anomaly coefficients to order $o(l_s^4/l_c^4)$ read (see e.g. [6]):

$$
\begin{aligned}
\beta^\Phi &= \frac{D-26}{6} + l_s^2\frac{(\nabla\Phi)^2+\nabla^2\Phi}{4} + l_s^4 R_{ABCD}R^{ABCD} = \frac{l_s^4}{l_c^4}\frac{5}{32g^4} \\
\beta^G_{\mu\nu} &= l_s^2(R_{\mu\nu} - \nabla_\mu\nabla_\nu\Phi) + l_s^4\frac{1}{2}R_{\mu ABC}R_\nu^{ABC} = \frac{l_s^4}{l_c^4}\frac{1}{4g^3}\eta_{\mu\nu} \\
\beta^G_{44} &= l_s^2(R_{44} - \nabla_4\nabla_4\Phi) + l_s^4\frac{1}{2}R_{4ABC}R_4^{ABC} = \frac{l_s^4}{l_c^2}\frac{1}{4g^4}
\end{aligned}
\tag{2.2}
$$

where $A, B, C, \ldots = 0\ldots 25$, $\mu, \nu\ldots = 0\ldots 3$, and $x^4 \equiv g$ denotes the holographic coordinate.

At this point it is worth mentioning that all the results obtained are independent of the choice of the metric signature.

The simplest way to find a background that enforces the vanishing of the beta functions to $o(l_s^4)$, starting with the unperturbed equation (1.1) and compatible with Poincaré invariance, consists of modifying the term conformal to the Minkowski metric in our background by inserting an arbitrary function $f(g)$:

$$
ds^2 = \left(g + \frac{l_s^2}{l_c^2}f(g)\right)\eta_{\mu\nu}dx^\mu dx^\nu + l_c^2 dg^2 + \sum_{A=5}^{A=26}(dx^A)^2\,. \tag{2.3}
$$

Note that in this way our dilaton remains unperturbed.

The condition we get for the vanishing of the beta functions to $o(l_s^4)$ is the following:

$$
\begin{aligned}
\beta^\phi &= \left[\frac{-f'}{2g^2} + \frac{f}{2g^3} + \frac{5}{32g^4}\right]\frac{l_s^4}{l_c^4} \\
\beta^G_{\mu\nu} &= \left[\frac{f}{2g^2} - \frac{f'}{2g} - \frac{f''}{2} + \frac{1}{4l_c^2 g^3}\right]\frac{l_s^4}{l_c^4}\eta_{\mu\nu} \\
\beta^G_{44} &= \left[\frac{-2f}{g^3} + \frac{2f'}{g^2} - \frac{2f''}{g} + \frac{1}{4l_c^2 g^4}\right]\frac{l_s^4}{l_c^2}\,.
\end{aligned}
\tag{2.4}
$$

Unfortunately the system above is algebraically incompatible .

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We are then forced to make a more general perturbation on the metric, inserting two new functions $f(g)$ and $w(g)$ in the expression:

$$ds^2 = \left(g + \frac{l_s^2}{l_c^2} f(g)\right) \eta_{\mu\nu} dx^\mu dx^\nu + l_c^2 \left(1 + \frac{l_s^2}{l_c^2} w(g)\right) dg^2 + \sum_{A=5}^{A=26} (dx^A)^2 \tag{2.5}$$

but keeping (2.1) for the dilaton.

It is trivial that this is equivalent, up to a change of coordinates, to a background with $w(g) = 0$, and a different $f'(g)$, as well as a new dilaton. In terms of the function $h(g)$, such that $dh/dg = w(g)$, the new dilaton is given by:[1]

$$\Phi = -\log(g) + \frac{l_s^2}{l_c^2} D(g) \tag{2.7}$$

where $D(g) = h(g)/2g$, and the new function $f'(g)$ is given in terms of the old one by:

$$f'(g) \equiv f(g) - \frac{h(g)}{2}\,. \tag{2.8}$$

The new beta functions read:

$$\begin{aligned}
\beta^\Phi &= \frac{l_s^4}{4 l_c^4} \left(\frac{w'}{2g} + \frac{2f}{g^3} - \frac{2f'}{g^2} + \frac{5}{8g^4}\right) \\
\beta^G_{\mu\nu} &= \frac{l_s^4}{l_c^4} \left(\frac{w'}{4} + \frac{f}{2g^2} - \frac{f'}{2g} - \frac{f''}{2} + \frac{1}{4g^3}\right) \eta_{\mu\nu} \\
\beta^G_{44} &= \frac{l_s^4}{l_c^2} \left(\frac{w'}{2g} - \frac{2f}{g^3} + \frac{2f'}{g^2} - \frac{2f''}{g} + \frac{1}{4g^4}\right)
\end{aligned} \tag{2.9}$$

and they vanish when:

$$\begin{aligned}
f(g) &= \left(c_1 g - \frac{1}{32g}\right) \\
w(g) &= \left(c_2 + \frac{1}{2g^2}\right).
\end{aligned} \tag{2.10}$$

Using now the relation (2.7) we may see this solution in an equivalent way as a change on the dilaton field:

$$D(g) = \frac{c_2}{2} + \frac{c_3}{2g} - \frac{1}{4g^2}\,. \tag{2.11}$$

Note that, in this manner, a new arbitrary constant c_3 arises. However, in this case, c_2 is nothing but the freedom we have to choose a zero point value on the original dilaton, due to the fact that it only appears in the beta equations in the form of a derivative.

[1] The change of coordinates is given by

$$g \longrightarrow g + \frac{l_s^2 h(g)}{2 l_c^2}\,. \tag{2.6}$$

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It is perhaps worth stressing that we have verified in passing that the only possible backgrounds $G_{\mu\nu}$, $B_{\mu\nu}$ preserving Poincaré invariance [2], which, together with our dilaton $\Phi = -\log g$, saturate the beta equations to order $o(l_s^2/l_c^2)$ are of the form:

$$ds^2 = A_1 g \eta_{\mu\nu} dx^\mu dx^\nu + A_2 l_c^2 dg^2 + \sum_{A=5}^{A=26} (dx^A)^2 \tag{2.12}$$

A_1, A_2 being arbitrary constants, whereas the Kalb-Ramond field has to be trivial, $H = 0$.

It is easy to see that this family of solutions is equivalent to our initial metric up to a scale transformation on the coordinate g as well as the addition of a proper constant to our original dilaton.

3. The fate of the singularity

Let us now consider the perturbed metric by itself:

$$\begin{aligned} ds^2 = &\left(g + \left(\frac{l_s}{l_c}\right)^2 \left(c_1 g - \frac{1}{32g} \right) \right) \eta_{\mu\nu} dx^\mu dx^\nu + \\ &+ l_c^2 \left(1 + \left(\frac{l_s}{l_c}\right)^2 \left(c_2 + \frac{1}{2g^2} \right) \right) dg^2 + \sum_{A=5}^{A=26} (dx^A)^2 . \end{aligned} \tag{3.1}$$

We are now interested in searching for the possible singularities of the metric that may have arisen after the $o(l_s^2/l_c^2)$ analytic perturbation.

As usual, we find them by imposing the vanishing of the determinant of either the metric or its inverse. From $det\, g_{AB}$=0 we get:

$$g_1^2 = l_s^2 \frac{1}{32(l_c^2 + l_s^2 c_1)} \tag{3.2}$$

$$g_2^2 = l_s^2 \frac{-1}{2(l_c^2 + l_s^2 c_2)} . \tag{3.3}$$

From $det\, g^{AB}$=0 we obtain the original singularity $g_3^2 = 0$.

How reliable are these putative singularities? First of all, in order for the whole perturbative expension to make sense, as we already said in the introduction, the curvature has to be smaller than the string scale. This gives a condition for the coordinate, namely,

$$g \gg \frac{l_s}{l_c} . \tag{3.4}$$

On the other hand, the curvature scalar of the new metric can be easily shown to diverge when

$$\left(-1 + \frac{g^2}{g_1^2} \right) \left(1 - \frac{g^2}{g_2^2} \right) = 0 , \tag{3.5}$$

[2] Which in the case of the Kalb-Ramond field (considered as a two-form) means that $\pounds(k)B = dC$.

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which clearly means that both $g = g_1$ and $g = g_2$ are true singularities; whereas $g = 0$ would rather become a Killing horizon.

There are now two possibilities: if both $(1 + c_2 l_s^2/l_c^2) > 0$ and $(1 + c_1 l_s^2/l_c^2) < 0$ then g_1 as well as g_2 are imaginary, so that the singularity has now been replaced by a horizon.

It is worth remarking, however, that for this to be true, the constant c_1 has to be large, of order $o(1/\epsilon)$, where ϵ is the parameter of the perturbation, $\epsilon = l_s^2/l_c^2$, which seems unnatural, although mathematically consistent.

On the other hand, if $(1 + c_1 l_s^2/l_c^2) < 0$ there is a change of signature in the part of the metric conformal to Minkowski space. If at least one of the above inequalities fails to be satisfied, then there is a singularity at some positive value of g, and the would-be horizon remains hidden beyond the said singularity. The only effect of the perturbation has then been to shift the singularity a little bit on the real axis.

The new metric could be written in a form quite similar to the lowest order one, by defining the new variable [3]

$$\tilde{g} \equiv g\sqrt{1 + \frac{l_s^2}{l_c^2}\left(c_2 + \frac{1}{2g^2}\right)} - \frac{1}{\sqrt{2}}\sqrt{\frac{l_s^2}{l_c^2}}\log\left(\frac{1}{g\sqrt{2l_s^2/l_c^2}} + \frac{\sqrt{1 + \frac{l_s^2}{l_c^2}\left(c_2 + 1/2g^2\right)}}{l_s^2/l_c^2}\right). \tag{3.6}$$

This yields the metric in the form

$$ds^2 = f(\tilde{g})\eta_{\mu\nu}dx^\mu dx^\nu + (d\tilde{g})^2 + \sum_{A=5}^{A=26}(dx^A)^2 \tag{3.7}$$

with an adequate $f(\tilde{g})$. These coordinates are useful only insofar as we restrict our interest to purely geometrical properties of the background, because in them the dilaton gets also modified, so that in general it is preferable to stick to the old system of coordinates (in which the dilaton is *universal*).

It is curious to observe that the Weyl tensor of the perturbed solution continues to vanish.

4. Conclusions

We have calculated the corrections to the sigma model equations (equivalent to demanding vanishing Weyl anomaly coefficients) to order $o(l_s^4)$.

The first result is that, modulo convenient (and algebraically complicated) changes of coordinates (of the type of the one indicated in (3.6) above), the metric can always be put in the general Poincaré invariant form (1.2), for which most convergence theorems have been proved, and which is conformally flat (its Weyl tensor vanishes).

[3] Determined from the condition that $(1 + (l_s/l_c)^2(c_2 + 1/2g^2))dg^2 = d\tilde{g}^2$.

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This presumably means that many of the general results on the gravitational (holographic) interpretation of the renormalization group, such as the c-theorem, etc. [1] when expressed in terms of the conformal factor $a(\rho)$ will remain valid, even after higher terms in the sigma model expansion are considered.

For generic values of the free parameters (integration constants) further results are not very spectacular; the position of the singularity is simply shifted a tiny amount (proportional to the perturbation) on the real axis. But for exceptional values of the parameters (namely, much bigger that the dimensionless strength of the perturbation) the singularity dissapears altogether and is replaced by a horizon.

Further computations are necessary before the physical reliability of the new horizon can be assessed.

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A. Low dimension coset models with Poincaré isometry group

It is well known that GKO coset models [4] can be generally described by a gauged WZW model with group G, of which only the (vector action of the) subgroup $H \subset G$ has been gauged [5]. Besides, in the semiclassical regime (that is, when $k \to \infty$) , the gauge fields themselves (which appear quadratically only) can be integrated, yielding in this way an ordinary sigma model, in a target space with dimension $d_{G/H} \equiv d_G - d_H$.

In order for a given transformation to appear as a isometry in the semiclassical sigma model, its action of the gauge fields must be trivial.

If we want, for example, to obtain a GKO coset, G/H with Poincaré invariance, then in order for $\mathrm{ISO}(3,1) \subset G$ to survive the vector gauging is necessary that $\mathrm{ISO}(3,1) \subset \mathcal{Z}[H]$ (the centralizer of H in G). But this implies that $H \subset \mathcal{Z}[\mathrm{ISO}(3,1)]$ and, given the fact that $\mathrm{ISO}(3,1)$ has no center, that H has to be embedded in a subgroup generated by a basis $B \subset \mathfrak{g}$ out of $\mathfrak{iso}(3,1)$.

The scenario of lowest dimension (leaving aside the simplest direct products of the form $(\mathrm{ISO}(3,1) \otimes H)/H$, with $H = \mathrm{U}(1)$ consists of building G with Poincaré generators plus two extra ones, say $\{Z, E\}$, Z conmuting with all Poincaré generators but not with E. Then we factor by $H = \mathrm{U}(1)$ embedded in the U(1) generated by Z.

Independently of the different possibilities of selecting the conmutation relations of E with $\mathfrak{iso}(3,1)$ algebra (compatible whith Jacobi identities), we expect the resulting coset to have dimension 11, provided that H is not a null subgroup of G (in this case an equivalence between vector and chiral gauge occurs accompained by an unexpected dimensional reduction of the GKO [3]).

In particular this excludes the possibility of finding a five-dimensional GKO with Poincaré invariance (in fact, we can not find a (nontrivial) d dimensional GKO with a symmetry group of dimension greater than $d-1$ provided the later has no center).

If we consider for example the group as above generated by

$$\{J_1, J_2, J_3, K_1, K_2, K_3, P_0, P_1, P_2, P_3, Z, E\} \tag{A.1}$$

(with J_i generators of rotations, K_i boosts , P_i translations[4] and the extra generators Z,E), then there either E itself (besides Z) does not appear in the commutator $[E, Z]$ or it does . In the first case, consistency with the Jacobi identities forces the subset $\{Z, E\}$ to be a subalgebra: $[Z, E] = \alpha Z$, $\alpha \neq 0$ the conmutation relations of E and Z with $\mathfrak{iso}(3,1)$ given by:

$$\begin{aligned}
[J_i, E] &= A_{ij} J_j + B_{ij} P_j \\
[K_i, E] &= A_{ij} K_j + \frac{\epsilon_{ijk}}{2} B_{jk} P_0 \\
[P_i, E] &= A_{ij} P_j \\
[Z, E] &= \alpha Z
\end{aligned}$$

(where $A_{(ij)} = B_{(ij)} = 0$). But performing the redefinition:

$$E' = E - \frac{i}{2}\epsilon_{nml} B_{ml} P_n - \frac{i}{2}\epsilon_{nml} A_{ml} J_n$$

it can be seen that this is nothing but the direct sum of $\mathfrak{iso}(3,1)$ and the solvable algebra spanned by $\{Z, E'\}$.

In the second case, i.e. when E belongs to the commutator of $[Z, E]$, it can be shown that the subset $\{Z, E\}$ is not a subalgebra. The conmutation relations above keep the same values, but new structure constants show up:

$$[E, Z] = \alpha Z + \beta E - \frac{i}{2}\beta\epsilon_{nml} A_{ml} J_n - \frac{i}{2}\beta\epsilon_{nml} B_{ml} Pn\,.$$

However, with the redefinition:

$$E' = E - \frac{i}{2}\epsilon_{nml} A_{ml} J_n - \frac{i}{2}\epsilon_{nml} B_{ml} P_n$$

the full algebra can be written again as $\mathfrak{iso}(3,1) \oplus \{Z, E'\}$as in the first case.

[4]To be specific, the Poincaré algebra in this basis is given by:

$$\begin{aligned}
&[J_i, J_j] = i\epsilon_{ijk} J_k \qquad &[K_i, K_j] = -i\epsilon_{ijk} K_k \qquad &[J_j, K_j] = i\epsilon_{ijk} K_k \\
&[J_i, P_j] = i\epsilon_{ijk} P_k \qquad &[K_i, P_0] = -iP_i \qquad &[K_i, P_j] = -i\delta_{ij} P_0\,.
\end{aligned} \tag{A.2}$$

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